The Response of a Thin Cylindrical Shell to Transient Surface Loading

JAMES SHENG*
North American Aviation, Inc., Downey, Calif.

Solutions are presented in explicit form for elastic flexural displacements and stresses of a simply supported, thin circular cylindrical shell of finite length subjected to transient surface loading. Williams' method, introduced here for the first time as applicable to the vibration problem of a shell, is utilized in the analytical treatment based on Donnell-type equations. The method differs from that of ordinary normal-mode solutions in that the response is considered in two constituent portions, namely, the static portion obtainable in a closed form by direct integration, and the remaining dynamic portion, expressible in series involving terms of normal modes of vibration. The advantage of this approach is in the convergence-accelerating ability of the series for the dynamic portion during the time when the transient load is being applied. The analysis is general in that it may be used to solve either transient or steady surface loading problems. A numerical example is employed for illustration.

Introduction

IN space as well as in the air, the shell structure of a spacecraft or a missile may encounter along its course nuclear blast, fuel explosion, gust, shock wave, or other momentary surface loading. Among all the geometrical configurations, the circular cylindrical shell is most commonly used as the major component part of the structure. The problem in structural dynamics is the prediction of the response of the shell to any of the transient loadings, and the damaging effect on the shell produced by such a loading.

The problems of free vibrations of a thin cylindrical shell, either of infinite or finite length, have aroused the interest of scientific societies for almost a century. A number of early papers and books such as those by Lord Rayleigh,[1-3] Love,[4, 5] and Basset[6] may be cited. The frequency equation for the general case was obtained, while the extensional and inextensional modes, as well as their exactness in the vicinity of the edges, were discussed.

Thereafter, Flügge[7] derived a slightly different set of equations of motion, and inferred from a numerical example that the extensional and inextensional modes do coexist. Using Lagrange's equations, Arnold and Warburton[8, 9] derived another set of equations and solved the free vibration problems of thin cylindrical shells having finite lengths. Similarly, Baron and Bleich,[10] employing Hamilton's principle, made corrections to Rayleigh's extensional modes for infinitely long, thin cylindrical shells. In 1955, Yu[11] initiated the use of Donnell-type equations[12] for free vibration problems.

On the other hand, the literature on the vibration problems of shell structures under transient loading is rather meager. In the present space age, problems such as these not only provide academic interest, but also are significant in application.

Presented as Preprint 64-484 at the AIAA 1st Annual Meeting, Washington, D. C., June 29–July 2, 1964; revision received January 11, 1965. This paper is partly based on work performed when the author was with the Martin Company. Permission of that company and particularly of Kenneth Jarmolow, former Manager of Martin's Department of Research and Development, for the author to include this work is gratefully acknowledged. Appreciation is also expressed to Francis C. Hung, Director of Dynamic Sciences, for his constant encouragement.

* Research Specialist, Dynamic Sciences, Space and Information Systems Division. Member AIAA.

Equations of Motion

Figure 1 shows a sketch and notations such as l for the axial length, a for the radius of the undeformed median surface of the cylindrical shell, h for the thickness, as well as sign conventions for the axial, the circumferential, and the radial coordinates, namely, x, θ, and z, respectively, and their corresponding displacements, u^*, v^*, and w^*.

Donnell-type equations of motion as used by Yu, characterized by their completely uncoupled form, are chosen for their simplicity and for the reasonable accuracy obtained in determining the natural frequencies, as compared with the results obtained with slightly different equations.[13] The Donnell-type equations are shown below:

$$\left.\begin{aligned}
&u_{xx}{}^* + \frac{1-\nu}{2a^2}u_{\theta\theta}{}^* + \frac{1+\nu}{2a}v_{x\theta}{}^* - \frac{\nu}{a}w_x{}^* = (1-\nu^2)\left(\frac{\rho}{E}\right)u_{tt}{}^* \\
&\frac{1+\nu}{2a}u_{x\theta}{}^* + \frac{1-\nu}{2}v_{xx}{}^* + \frac{1}{a^2}v_{\theta\theta}{}^* - \frac{1}{a^2}w_\theta{}^* = (1-\nu^2)\left(\frac{\rho}{E}\right)v_{tt}{}^* \\
&\frac{\nu}{a}u_x{}^* + \frac{1}{a^2}v_\theta{}^* - \frac{w^*}{a^2} - \left(\frac{h^2}{12}\right)\nabla^*w^* = (1-\nu^2)\left(\frac{\rho}{E}\right)\left(w_{tt}{}^* - \frac{q^*}{h\rho}\right)
\end{aligned}\right\} \quad (1)$$

where the subscripts x, θ, and t indicate partial differentiation

$$\nabla^* = [(\partial^2/\partial x^2) + (\partial^2/a^2\partial\theta^2)]^2$$

q^* is the transient or steady surface loading per unit area, and t is the time. The material properties are represented in terms of Young's modulus E, mass density ρ, and Poisson's ratio ν.

Separation of Variables

Any surface loading at any time can be resolved into two constituent portions: the symmetric and the antisymmetric ones with respect to $\theta = 0$. The total displacements, strains, and stresses may be obtained by superposition of these two sets of the results.

If the symmetric loading q^* is nondimensionalized by the representation

$$q^*(x, \theta, t) = \frac{aD}{l^4} \sum_{m=0}^{\infty} q_m(x, t) \cos m\theta \tag{2}$$

where the flexural rigidity $D = Eh^3/12\ (1 - \nu^2)$, the appearance of Eq. (1) suggests a separation of the variable θ from the others[11, 14] employing the following substitutions:

$$\left.\begin{aligned} \frac{u^*}{a} &= \sum_m u_m(x, t) \cos m\theta \\ \frac{v^*}{a} &= \sum_m v_m(x, t) \sin m\theta \\ \frac{w^*}{a} &= \sum_m w_m(x, t) \cos m\theta \end{aligned}\right\} \tag{3}$$

where m is an integer and may be interpreted as the number of the circumferential waves around the cylindrical shell.

For the case of antisymmetric loading,[15] the final expressions for the displacements and stresses may be readily obtained by the replacement of $\sin m\theta$ for $\cos m\theta$ and $-\cos m\theta$ for $\sin m\theta$ in those for the case of the symmetric loading. Consequently, only the latter case will be fully treated in this paper.

The nondimensional displacements u_m, v_m, and w_m related to the mth harmonic of θ may be expressed as functions of the nondimensional axial coordinate ξ equal to x/l and the nondimensional time τ, which is defined in the ratio

$$\frac{\tau}{t} = \frac{\omega_i}{k_i} = \frac{1}{l} \left(\frac{E/\rho}{1 - \nu^2} \right)^{1/2}$$

where the ith natural circular frequency ω_i is also nondimensionalized to k_i. This ratio may be called a similarity scale. When this similarity scale is employed, results for a cylindrical shell can be applied to another with the same geometrical ratios but of different material properties.

With all the substitutions, and if the subscripts m are dropped from u_m, v_m, w_m, and q_m for simplicity, the following nondimensional θ-free Donnell-type equations of motion are found to be

$$\left.\begin{aligned} & u'' - \frac{1-\nu}{2} m^2 \left(\frac{l}{a}\right)^2 u + \frac{1+\nu}{2} m \left(\frac{l}{a}\right) v' - \nu \left(\frac{l}{a}\right) w' = \ddot{u} \\ & \frac{1-\nu}{2} v'' - m^2 \left(\frac{l}{a}\right)^2 v - \frac{1+\nu}{2} m \left(\frac{l}{a}\right) u' + m \left(\frac{l}{a}\right)^2 w = \ddot{v} \\ & \frac{\nabla^4 w}{K} + \left(\frac{l}{a}\right)^2 w = \frac{q}{K} - \ddot{w} + \nu \left(\frac{l}{a}\right) u' + m \left(\frac{l}{a}\right)^2 v \end{aligned}\right\} \tag{4}$$

where

$$\nabla^4 = \nabla^2\nabla^2 = [(\partial^2/\partial\xi^2) - m^2(l/a)^2]^2 \qquad K = 12(l/h)^2$$

The prime indicates partial differentiation with respect to ξ and the dot indicates partial differentiation with respect to τ.

Boundary and Initial Conditions

The boundary conditions for a simply-supported, circular cylindrical shell are such that the radial and circumferential displacements, the axial membrane stresses, and the axial moment-resultants at the boundaries are zero at all times. If the relations for the stresses and moment-resultants are expressed as functions of displacements,[14, 15] the following equivalent boundary conditions can be deduced:

$$\left.\begin{aligned} w(0, \tau) &= w(1, \tau) = 0 \\ w''(0, \tau) &= w''(1, \tau) = 0 \\ u'(0, \tau) &= u'(1, \tau) = 0 \\ v(0, \tau) &= v(1, \tau) = 0 \end{aligned}\right\} \tag{5}$$

where 0 and 1 inside the parentheses indicate the values of ξ.

The cylindrical shell is assumed initially at rest and undeflected; hence, the initial conditions for all values of ξ, i.e., $0 \leq \xi \leq 1$, are

$$\left.\begin{aligned} w(\xi, 0) &= \dot{w}(\xi, 0) = 0 \\ u(\xi, 0) &= \dot{u}(\xi, 0) = 0 \\ v(\xi, 0) &= \dot{v}(\xi, 0) = 0 \end{aligned}\right\} \tag{6}$$

Williams' Method

In normal-mode solutions for transient load problems, the response is expanded in terms of a series of normal modes. The solutions arrived at by means of Williams' method[16–18] differ from ordinary normal-mode solutions by virtue of the decomposition of the response into two portions: the static portion obtainable in a closed form by a process of direct integration, and the remaining dynamic portion expressible in series form involving terms of normal modes of vibration.

There is an advantage in the Williams' method over that of ordinary modal solutions in the convergence-accelerating ability of the series for the dynamic portion of the former during the time when the transient load is being applied; more accurate results are obtained with the same number of terms as in the series used in the latter method. Williams' method was discussed favorably by Ramberg,[19] for application to analysis of transient vibration of an airplane wing. Not too long ago Leonard[20] employed it to find the solutions for the transient response of beams, and Sheng[21] extended it to the problems of shell structures subjected to radiation shock.

Static and Natural-Mode Displacements

Based on Williams' method, the displacements can be decomposed into their static and dynamic or natural-mode components. The solutions for Eqs. (4) are assumed in the form

$$u(\xi, \tau) = u_s(\xi, \tau) + \sum_i \Phi_i(\tau) u_i(\xi) \tag{7a}$$

$$v(\xi, \tau) = v_s(\xi, \tau) + \sum_i \Phi_i(\tau) v_i(\xi) \tag{7b}$$

$$w(\xi, \tau) = w_s(\xi, \tau) + \sum_i \Phi_i(\tau) w_i(\xi) \tag{7c}$$

where $\Phi_i(\tau)$ is called the generalized coordinate, the subscripts s indicate the static displacements, whereas the subscripts i relate to one of the frequencies of the natural modes of vibration. This representation gives rise to two sets of boundary conditions and equations: static and natural-mode ones.

New Sets of Boundary Conditions

Equations (7) suggest two new sets of the boundary conditions, comprising those for static displacements

$$w_s(0, \tau) = w_s(1, \tau) = w_s''(0, \tau) = w_s''(1, \tau) = 0 \tag{8a}$$

$$u_s'(0, \tau) = u_s'(1, \tau) = v_s(0, \tau) = v_s(1, \tau) = 0 \tag{8b}$$

and those for the natural-mode displacements deduced from Eqs. (5, 7, and 8), which can be expressed in the form

$$w_i(0) = w_i(1) = w_i''(0) = w_i''(1) = 0 \qquad (9a)$$

$$u_i'(0) = u_i'(1) = v_i(0) = v_i(1) = 0 \qquad (9b)$$

With these boundary conditions, the two portions of the solutions will be discussed separately.

Static and Natural-Mode Equations

In the absence of the inertia forces, from Eqs. (4) and (7) the following static equations may be obtained:

$$u_s'' - \frac{1-\nu}{2} m^2 \left(\frac{l}{a}\right)^2 u_s + \frac{1+\nu}{2} m \left(\frac{l}{a}\right) v_s' - \nu \left(\frac{l}{a}\right) w_s' = 0 \qquad (10a)$$

$$\frac{1-\nu}{2} v_s'' - m^2 \left(\frac{l}{a}\right)^2 v_s - \frac{1+\nu}{2} m \left(\frac{l}{a}\right) u_s' + m \left(\frac{l}{a}\right)^2 w_s = 0 \qquad (10b)$$

$$\nabla^4 w_s + K \left(\frac{l}{a}\right)^2 w_s = q + K \left[\nu \left(\frac{l}{a}\right) u_s' + m \left(\frac{l}{a}\right)^2 v_s\right] \qquad (10c)$$

As the displacements of each natural mode are taken to be harmonic functions of τ, the natural-mode equations may be represented as:

$$u_i'' - \frac{1-\nu}{2} m^2 \left(\frac{l}{a}\right)^2 u_i + \frac{1+\nu}{2} m \left(\frac{l}{a}\right) v_i' - \nu \left(\frac{l}{a}\right) w_i' + k_i^2 u_i = 0 \qquad (11a)$$

$$\frac{1-\nu}{2} v_i'' - m^2 \left(\frac{l}{a}\right)^2 v_i - \frac{1+\nu}{2} m \left(\frac{l}{a}\right) u_i' + m \left(\frac{l}{a}\right)^2 w_i + k_i^2 v_i = 0 \qquad (11b)$$

$$\nabla^4 w_i + K \left(\frac{l}{a}\right)^2 w_i - K \left[\nu \left(\frac{l}{a}\right) u_i' + m \left(\frac{l}{a}\right)^2 v_i\right] = K k_i^2 w_i \qquad (11c)$$

If u, v, and w in Eqs. (4) are replaced with Eqs. (7), and then Eqs. (10) and (11) are substituted in the resulting equations, the coupled relationship between the static and the dynamic portions of the displacements can be seen in the following:

$$\left.\begin{aligned} \sum_i [\ddot{\Phi}_i(\tau) + k_i^2 \Phi_i(\tau)] w_i(\xi) &= -w_s(\xi, \tau) \\ \sum_i [\ddot{\Phi}_i(\tau) + k_i^2 \Phi_i(\tau)] u_i(\xi) &= -\ddot{u}_s(\xi, \tau) \\ \sum_i [\ddot{\Phi}_i(\tau) + k_i^2 \Phi_i(\tau)] v_i(\xi) &= -\ddot{v}_s(\xi, \tau) \end{aligned}\right\} \qquad (12)$$

Orthogonality Condition

To obtain equations for determining the individual function Φ_i, the concept of orthogonality of the eigenfunctions will be employed. When Eqs. (11) are used, a subscript j different from the subscript i is assumed; the orthogonality condition

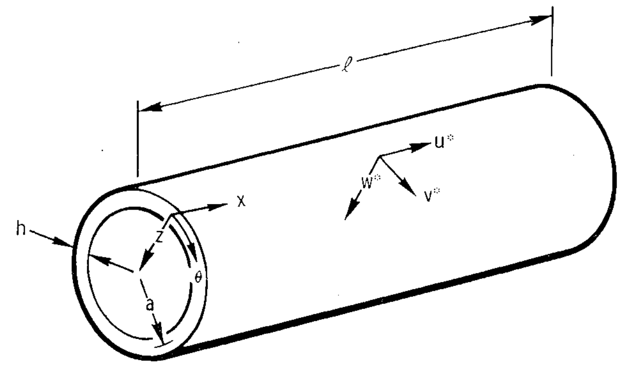

Fig. 1 Sign conventions for coordinates and displacements.

$$\int_0^1 (w_i w_j + v_i v_j + u_i u_j) d\xi = 0 \qquad (i \neq j) \qquad (13)$$

may be established after integration by parts is carried out with the aid of the boundary conditions (9).

Generalized Force and Generalized Mass

Multiplying Eqs. (12) by w_i, u_i, and v_i, respectively, integrating and adding the results, one obtains the equation

$$\ddot{\Phi}_i(\tau) + k_i^2 \Phi_i(\tau) = -[\ddot{P}_i(\tau)/m_i] \qquad (14)$$

in which the generalized force

$$P_i(\tau) = \int_0^1 (u_i u_s + v_i v_s + w_i w_s) d\xi \qquad (15)$$

and the generalized mass

$$m_i = \int_0^1 (u_i^2 + v_i^2 + w_i^2) d\xi \qquad (16)$$

With the substitution of the initial conditions (6) into expressions (7) and the use of Eqs. (15) and (16), a new set of initial conditions for Eq. (14) may be expressed as

$$\begin{aligned} \Phi_i(0) &= -[P_i(0)/m_i] \\ \dot{\Phi}_i(0) &= -[\dot{P}_i(0)/m_i] \end{aligned} \qquad (17)$$

Insertion of Eqs. (11) for u_i, v_i, and w_i into Eq. (15), successive integration of the resulting equation by parts taking into account boundary conditions (8) and (9), and the substitution of Eqs. (10) in the final equation yield

$$P_i(\tau) = \frac{1}{K k_i^2} \int_0^1 q(\xi, \tau) w_i(\xi) d\xi \qquad (18)$$

The presence of k_i^2 manifests the more rapid convergence of Williams' method as compared with that of ordinary modal solutions mentioned earlier.

Generalized Coordinate

If the Laplace transform is used, the solution of Eq. (14) for the ith generalized coordinate $\Phi_i(\tau)$ may be represented as

$$\Phi_i(\tau) = \bar{A} \sin k_i \tau + \bar{B} \cos k_i \tau - \frac{1}{m_i k_i} \int_0^\tau \sin k_i(\tau - \varphi) \frac{d^2 P_i(\varphi)}{d\varphi^2} d\varphi \qquad (19)$$

where $\bar{A}$ and $\bar{B}$ are arbitrary constants and φ is a dummy variable.

Repeating integration of Eq. (19) by parts, using the technique for the differentiation of an integral with respect to a parameter, and comparing the resulting equation for the generalized coordinate and its first derivative for $\tau = 0$ with Eqs. (17), one concludes

$$\bar{A} = -\dot{P}_i(0)/m_ik_i \qquad \bar{B} = -P_i(0)/m_i \tag{20}$$

As a consequence, the formula for the generalized coordinate becomes

$$\Phi_i(\tau) = -\frac{P_i(\tau)}{m_i} + \frac{k_i}{m_i}\int_0^\tau P_i(\varphi)\sin k_i(\tau - \varphi)d\varphi \tag{21}$$

Solutions of Natural-Mode Equations

Rearranging Eqs. (11a) and (11b) by differentiation and elimination in a manner similar to that employed by Donnell,[12] one obtains

$$\bar{\nabla} u_i = \left(\frac{l}{a}\right)\left\{\nu(1-\nu)w_i''' + \left[2\nu k_i^2 + (1-\nu)m^2\left(\frac{l}{a}\right)^2\right]w_i'\right\} \tag{22a}$$

$$\bar{\nabla} v_i = m\left(\frac{l}{a}\right)^2\left\{\left[(1-\nu)m^2\left(\frac{l}{a}\right)^2 - 2k_i^2\right]w_i - (1-\nu)(2+\nu)w_i''\right\} \tag{22b}$$

in which

$$\bar{\nabla} = [(1-\nu)\nabla^2 + 2k_i^2](\nabla^2 + k_i^2)$$

Application of the operator $\bar{\nabla}$ to Eq. (11c) and subsequent elimination of the terms containing u_i and v_i from the resulting expression with the aid of Eqs. (22a) and (22b) yield the following uncoupled homogeneous ordinary differential equation for the dependent variable w_i, for which the order is consequently raised:

$$\bar{\nabla}\left[\frac{1}{K}\nabla^4 + \left(\frac{l}{a}\right)^2 - k_i^2\right]w_i = \nu^2(1-\nu)\left(\frac{l}{a}\right)^2 w_i^{\mathrm{IV}} + 2\left(\frac{l}{a}\right)^2\left[\nu^2k_i^2 - (1-\nu)m^2\left(\frac{l}{a}\right)^2\right]w_i'' + m^2\left(\frac{l}{a}\right)^4\left[(1-\nu)m^2\left(\frac{l}{a}\right)^2 - 2k_i^2\right]w_i \tag{22c}$$

With boundary conditions (9) and additional ones obtainable from Eqs. (11) as well as their once and twice differentiated forms, it can be shown that the one and only nontrivial solution[22, 23] for $w_i(\xi)$ is

$$w_i(\xi) = C\sin n\pi\xi \tag{23a}$$

Table 1 Amplitude ratios of displacements for $l/a = 0.01$ and $m = 4$

Frequency	A/C	B/C
$n = 1$		
Low	-9.55×10^{-4}	9.34×10^{-5}
Medium	4.95×10^{-2}	4.00
High	-1.15	1.66×10^{-2}
$n = 4$		
Low	-2.39×10^{-4}	5.80×10^{-6}
Medium	-8.60×10^{-2}	-2.70×10
High	1.80	-5.72×10^{-3}
$n = 7$		
Low	-1.37×10^{-4}	1.91×10^{-6}
Medium	-2.10×10^{-4}	-4.00×10^{-2}
High	9.56×10^{-1}	-1.74×10^{-3}
$n = 10$		
Low	-9.59×10^{-5}	9.42×10^{-7}
Medium	-1.47×10^{-4}	-2.00×10^{-2}
High	-2.09×10^{-1}	2.66×10^{-4}

where C is an arbitrary constant and n may be interpreted as the number of half-waves along the generator of the cylindrical shell. Similarly, one may also obtain

$$u_i(\xi) = A\cos n\pi\xi \tag{23b}$$

$$v_i(\xi) = B\sin n\pi\xi \tag{23c}$$

where A and B are constants.

Natural Frequencies, Displacements, and Generalized Mass

When the notations

$$\Lambda = m^2(l/a)^2 + n^2\pi^2 \qquad \Omega = 2k_i^2/(1-\nu) \tag{24}$$

are used and the substitution of Eq. (23a) in Eq. (22c) is performed, the frequency equation may be expressed as

$$\Omega^3 - P\Omega^2 + Q\Omega - R = 0 \tag{25}$$

where

$$\left.\begin{aligned} P &= \frac{1}{1-\nu}\left\{2\left[\frac{\Lambda^2}{K} + \left(\frac{l}{a}\right)^2\right] + (3-\nu)\Lambda\right\} \\ Q &= \frac{2}{1-\nu}\left[\frac{3-\nu}{1-\nu}\frac{\Lambda^3}{K} + \Lambda^2 + \left(\frac{l}{a}\right)^2\Lambda + 2(1+\nu)n^2\pi^2\left(\frac{l}{a}\right)^2\right] \\ R &= \frac{4}{(1-\nu)^2}\left[\frac{\Lambda^4}{K} + (1-\nu^2)n^4\pi^4\left(\frac{l}{a}\right)^2\right] \end{aligned}\right\} \tag{25a}$$

It is apparent that for every m and n, there are three natural frequencies. Hence, the subscript i, used previously and hereafter, refers to the particular mode of vibration not only pertaining to the specific values of the integers m and n but also to that of one of the three natural frequencies.

In Eq. (23b), the constant A represents the maximum magnitude of the axial displacement of the ith mode; in Eq. (23c), the constant B represents that of its circumferential displacement; whereas in Eq. (23a), the constant C represents that of its radial displacement. The relative magnitudes of the displacements are readily obtained by substituting Eqs. (23a–23c) in Eqs. (11a) and (11b). The results are:

$$A/C = 2(l/a)n\pi(\alpha/\gamma\eta) \tag{26a}$$

$$B/C = 2(l/a)^2m(\beta/\gamma\eta) \tag{26b}$$

in which

$$\left.\begin{aligned} \alpha &= \Lambda - \Omega + (1+\nu)(\Omega - n^2\pi^2) \\ \beta &= \Lambda - \Omega + (1+\nu)n^2\pi^2 \\ \gamma &= \Lambda - \Omega \\ \eta &= \alpha + \beta = 2\Lambda - (1-\nu)\Omega \end{aligned}\right\} \tag{27}$$

Equations (26a) and (26b) indicate that whereas the magnitudes of the displacements for each mode of the free vibrations are arbitrary, their ratios are definite and dependent not only on m and n, but also on one of the three frequencies with which they are associated. In Ref. 8, one may find a representative figure established for $m = 4$ depicting the relationship between n and the frequencies.

The following statements of interest are also found in the same source. For the lowest frequencies, the vibratory motion is mainly radial. For the next higher family of frequencies, the axial motion is predominant when n is small, while the axial and the circumferential components become comparable for large values of n. There is little bending for this category. For the highest frequencies, the vibratory motion is predominantly circumferential, except at high values of n when the axial motion becomes comparable to that of the circumferential components.

Viewing Eqs. (26a) and (26b), one may recognize that for extremely small l/a, these characteristic magnitudes may be somewhat altered. Table 1, shown below, lists the results obtained for a cylindrical shell whose $l/a = 0.01$, $h/a = 6.462 \times 10^{-5}$, and $\nu = 0.3$ for the case of $m = 4$, which bears the same ratio of length to thickness as that in Table 2 of Ref. 8 so that a comparison can be made.

From Table 1 for the case when $l/a \ll 1$, one can conclude the following: for the lowest frequencies, the vibratory motion is mainly radial, only more so than the case when the length l is not so small as compared with the radius a; for the next higher family of frequencies, the circumferential motion is predominant when n is small, whereas the vibratory motion is chiefly radial when n is large; for the highest frequencies, the axial motion predominates for small n but it is comparable to the radial motion for large n.

When Eqs. (23a–23c) are substituted in Eq. (16), the generalized mass can be simplified to

$$m_i = \tfrac{1}{2}(A^2 + B^2 + C^2) \tag{28}$$

Further substitutions of Eqs. (26a) and (26b) in Eq. (28) lead to

$$m_i = \frac{1}{2}\left\{1 + \frac{4(l/a)^2[n^2\pi^2\alpha^2 + m^2(l/a)^2\beta^2]}{\gamma^2\eta^2}\right\} C^2 \tag{29}$$

This is the formula of the generalized mass for the particular m, n, and Ω, designated by the subscript i.

Static Displacements

Following Donnell's procedure, which has been used earlier in treating the equations of motion for natural modes, one obtains from Eqs. (10) the uncoupled static equations for displacements as in the following:

$$\nabla^4 u_s = (l/a)[(1+\nu)w_s''' - \nabla^2 w_s'] \tag{30a}$$

$$\nabla^4 v_s = -m(l/a)^2[(1+\nu)w_s'' + \nabla^2 w_s] \tag{30b}$$

$$\nabla^8 w_s + \bar{K} w_s^{\mathrm{IV}} = \nabla^4 q \tag{30c}$$

where

$$\bar{K} = (1-\nu^2)(l/a)^2 K$$

If the notations

$$\begin{aligned} \lambda_{1,2} &= \tfrac{1}{2}[(1+i)\mathcal{K} \pm (\bar{\alpha} + i\bar{\beta})] \\ \lambda_{3,4} &= \tfrac{1}{2}[-(1+i)\mathcal{K} \pm (\bar{\alpha} + i\bar{\beta})] \\ \lambda_{5,6} &= \tfrac{1}{2}[(1-i)\mathcal{K} \pm (\bar{\alpha} - i\bar{\beta})] \\ \lambda_{7,8} &= \tfrac{1}{2}[-(1-i)\mathcal{K} \pm (\bar{\alpha} - i\bar{\beta})] \end{aligned} \tag{31}$$

are used, in which

$$\left.\begin{aligned} \mathcal{K} &= (\bar{K}/4)^{1/4} \qquad i = (-1)^{1/2} \\ \bar{\alpha} &= \left\{\left[\frac{\bar{K}}{4} + 4m^4\left(\frac{l}{a}\right)^4\right]^{1/2} + 2m^2\left(\frac{l}{a}\right)^2\right\}^{1/2} \\ \bar{\beta} &= \left\{\left[\frac{\bar{K}}{4} + 4m^4\left(\frac{l}{a}\right)^4\right]^{1/2} - 2m^2\left(\frac{l}{a}\right)^2\right\}^{1/2} \end{aligned}\right\} \tag{32}$$

and since all the values of λ are distinct, the general solution for Eq. (30c) may be expressed as

$$w_s(\xi, \tau) = F(\xi, \tau) + \sum_{r=1}^{8} B_r(\tau) \exp\lambda_r\xi \tag{33}$$

of which the particular solution can be represented by

$$F(\xi, \tau) = \sum_{r=1}^{8} \frac{\lambda_r}{\prod_{s=1}^{8}[\lambda_r - \lambda_s(1-\delta_{rs})]} \times \int^{\xi} [\exp\lambda_r(\xi - \varphi)]\nabla^4 q(\varphi, \tau)d\varphi \tag{34}$$

where δ_{rs} is Kronecker delta, and φ is a dummy variable. The lower limit of integration may be arbitrarily fixed, since the term which proceeds from a constant lower limit of integration is included in the complementary part of the solution.

In order to solve Eqs. (33, 30a, and 30b), additional boundary conditions are required. They can be obtained by substitution and resubstitution of Eqs. (8) in Eqs. (10b), (10c), and equations of higher orders attained by differentiation of (10a) once and of (10c) twice with respect to ξ evaluated at $\xi = 0$ and $\xi = 1$. One may list them as follows:

$$\left.\begin{aligned} & w_s^{\mathrm{IV}}(0, \tau) = q(0, \tau) \qquad w_s^{\mathrm{IV}}(1, \tau) = q(1, \tau) \\ & w_s^{\mathrm{VI}}(0, \tau) = q''(0, \tau) + 2m^2\left(\frac{l}{a}\right)^2 q(0, \tau) \\ & w_s^{\mathrm{VI}}(1, \tau) = q''(1, \tau) + 2m^2\left(\frac{l}{a}\right)^2 q(1, \tau) \end{aligned}\right\} \tag{35a}$$

$$u_s'''(0, \tau) = u_s'''(1, \tau) = v_s''(0, \tau) = v_s''(1, \tau) = 0 \tag{35b}$$

Substitution of the boundary conditions, namely, Eqs. (35a) and (8a) in Eq. (33) and its derivatives of higher orders with respect to ξ at $\xi = 0$ and $\xi = 1$ gives the following matrix equation:

$$[A][B] = [C] \tag{36}$$

where

$$[A] = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \lambda_1^2 & \lambda_2^2 & \lambda_2^2 & \lambda_1^2 & \lambda_5^2 & \lambda_6^2 & \lambda_6^2 & \lambda_5^2 \\ \lambda_1^4 & \lambda_2^4 & \lambda_2^4 & \lambda_1^4 & \lambda_5^4 & \lambda_6^4 & \lambda_6^4 & \lambda_5^4 \\ \lambda_1^6 & \lambda_2^6 & \lambda_2^6 & \lambda_1^6 & \lambda_5^6 & \lambda_6^6 & \lambda_6^6 & \lambda_5^6 \\ e^{\lambda_1} & e^{\lambda_2} & e^{-\lambda_2} & e^{-\lambda_1} & e^{\lambda_5} & e^{\lambda_6} & e^{-\lambda_6} & e^{-\lambda_5} \\ \lambda_1^2 e^{\lambda_1} & \lambda_2^2 e^{\lambda_2} & \lambda_2^2 e^{-\lambda_2} & \lambda_1^2 e^{-\lambda_1} & \lambda_5^2 e^{\lambda_5} & \lambda_6^2 e^{\lambda_6} & \lambda_6^2 e^{-\lambda_6} & \lambda_5^2 e^{-\lambda_5} \\ \lambda_1^4 e^{\lambda_1} & \lambda_2^4 e^{\lambda_2} & \lambda_2^4 e^{-\lambda_2} & \lambda_1^4 e^{-\lambda_1} & \lambda_5^4 e^{\lambda_5} & \lambda_6^4 e^{\lambda_6} & \lambda_6^4 e^{-\lambda_6} & \lambda_6^4 e^{-\lambda_5} \\ \lambda_1^6 e^{\lambda_1} & \lambda_2^6 e^{\lambda_2} & \lambda_2^6 e^{-\lambda_2} & \lambda_1^6 e^{-\lambda_1} & \lambda^6{}_5 e^{\lambda_5} & \lambda_6^6 e^{\lambda_6} & \lambda_6^6 e^{-\lambda_6} & \lambda_5^6 e^{-\lambda_5} \end{bmatrix} \tag{36a}$$

and the elements of the column matrix C are

$$\left.\begin{aligned} & C_1 = -F(0, \tau) \qquad C_2 = -F''(0, \tau)/m^2(l/a)^2 \\ & C_3 = [q(0, \tau) - F^{\mathrm{IV}}(0, \tau)]/m^4(l/a)^4 \\ & C_4 = [q''(0, \tau) + 2m^2(l/a)^2 q(0, \tau) - F^{\mathrm{VI}}(0, \tau)]/m^6(l/a)^6 \\ & C_5 = -F(1, \tau) \qquad C_6 = -F''(1, \tau)/m^2(l/a)^2 \\ & C_7 = [q(1, \tau) - F^{\mathrm{IV}}(1, \tau)]/m^4(l/a)^4 \\ & C_8 = [q''(1, \tau) + 2m^2(l/a)^2 q(1, \tau) - F^{\mathrm{VI}}(1, \tau)]/m^6(l/a)^6 \end{aligned}\right\} \tag{37}$$

As a consequence, the nondimensional static radial displacement w_s can be obtained after the constants $B_1, B_2, \ldots, B_8$ are evaluated from Eq. (36).

Substituting the resulting equation for w_s in Eqs. (30a) and (30b) and integrating these with the help of boundary conditions (35b) and (8b) lead to two more matrix equations for the evaluation of u_s' and v_s. Since u_s can only be determined to within an arbitrary constant, one can specify that u_s be zero at some convenient location, say the midpoint of a generator.

Axisymmetric Case of Natural Modes

Until now, only the case for nonaxisymmetric modes of the natural vibration has been explored. As a matter of convenience, the axisymmetric case can be considered as that case in which $m = 0$ as well as $v = v_i = v_s \equiv 0$. This is true when one wants to convert Eqs. (10–13, 15, and 16) into those for the axisymmetric case. But Eqs. (14, 17, 18, 21, 23a, and 23b) appear in the same form. One may denote q_0 specifically instead of plain q for the nondimensional pressure loading associated with $m = 0$, as can be found in Eq. (2). Equation (26a) may be reduced to

$$\frac{A}{C} = \frac{\nu(l/a)n\pi}{k_i^2 - n^2\pi^2} \tag{38}$$

However, the frequency equation cannot be deduced from that for the nonaxisymmetric case, but needs to be derived from the natural-mode equations obtained for the axisymmetric case. The resulting frequencies may be expressed as

$$k_i^2 = \frac{1}{2}\left\{\left[\frac{n^4\pi^4}{K} + n^2\pi^2 + \left(\frac{l}{a}\right)^2\right] \pm \left(\left[\frac{n^4\pi^4}{K} - n^2\pi^2 + \left(\frac{l}{a}\right)^2\right]^2 + 4\nu^2\left(\frac{l}{a}\right)^2 n^2\pi^2\right)^{1/2}\right\} \tag{39}$$

It is evident that only two natural frequencies exist for each n for the axisymmetric case. Consequently, the subscript i relates to one of these two frequencies. There are no circumferential displacements. Ordinarily, for the higher of the two frequencies, the vibratory motion is mainly axial; for the lower one, the radial amplitude is predominant. However, the reverse may be true if the radius of the cylindrical shell is exceptionally large in comparison to its axial length as can be verified by examining the combined effect of Eqs. (38) and (39). Livanov[24] discussed this phenomenon for such a rare case based on a slightly different set of equations of motion by Vlasov.[25]

The generalized mass m_i for the axisymmetric case may easily be reduced to

$$m_i = \frac{1}{2}\left[1 + \frac{\nu^2(l/a)^2 n^2\pi^2}{(k_i^2 - n^2\pi^2)^2}\right] C^2 \tag{40}$$

Static Displacements for Axisymmetric Case

The equations of static displacements for the axisymmetric case are shown below:

$$u_s' = \nu(l/a)w_s \qquad w_s^{\mathrm{IV}} + \bar{K}w_s = q_0 \tag{41}$$

The second of Eqs. (41) can be used in solving for w_s along with boundary conditions (8a). Thus, if one denotes

$$\bar{\lambda}_1 = -\bar{\lambda}_3 = (1 + i)\mathcal{K} \qquad \bar{\lambda}_2 = -\bar{\lambda}_4 = (1 - i)\mathcal{K} \tag{42}$$

where i represents $(-1)^{1/2}$, then

$$w_s(\xi, \tau) = G(\xi, \tau) + \sum_{r=1}^{4} \bar{B}_r(\tau) \exp\bar{\lambda}_r\xi \tag{43}$$

in which $\bar{B}_1$, $\bar{B}_2$, $\bar{B}_3$, and $\bar{B}_4$ are arbitrary constants at any τ and $G(\xi, \tau)$ may be represented by

$$G(\xi, \tau) = \sum_{r=1}^{4} \frac{\bar{\lambda}_r}{\prod_{s=1}^{4} [\bar{\lambda}_r - \bar{\lambda}_s(1 - \delta_{rs})]} \times \int^{\xi} [\exp\bar{\lambda}_r(\xi - \varphi)] q_0(\varphi, \tau) d\varphi \tag{44}$$

where δ_{rs} is Kronecker delta, and φ is a dummy variable. The lower limit of integration may be arbitrarily fixed, for the term which proceeds from a constant lower limit of integration is included in the complementary part of the solution.

Applying boundary conditions (8a) to Eq. (43) and its second derivative with respect to ξ leads to

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ \bar{\lambda}_1^2 & \bar{\lambda}_2^2 & \bar{\lambda}_1^2 & \bar{\lambda}_2^2 \\ e^{\bar{\lambda}_1} & e^{\bar{\lambda}_2} & e^{-\bar{\lambda}_1} & e^{-\bar{\lambda}_2} \\ \bar{\lambda}_1^2 e^{\bar{\lambda}_1} & \bar{\lambda}_2^2 e^{\bar{\lambda}_2} & \bar{\lambda}_1^2 e^{-\bar{\lambda}_1} & \bar{\lambda}_2^2 e^{-\bar{\lambda}_2} \end{bmatrix} \begin{bmatrix} \bar{B}_1(\tau) \\ \bar{B}_2(\tau) \\ \bar{B}_3(\tau) \\ \bar{B}_4(\tau) \end{bmatrix} = -\begin{bmatrix} G(0, \tau) \\ G''(0, \tau) \\ G(1, \tau) \\ G''(1, \tau) \end{bmatrix} \tag{45}$$

If $q_0(\xi, \tau) = q_0(\tau)$, independent of ξ, then

$$w_s(\xi, \tau) = \frac{q_0(\tau)}{\bar{K}} \times \left\{1 - \frac{\cos\mathcal{K}(1 - \xi)\cosh\mathcal{K}\xi + \cos\mathcal{K}\xi\cosh\mathcal{K}(1 - \xi)}{\cos\mathcal{K} + \cosh\mathcal{K}}\right\} \tag{46}$$

which may be found in Ref. 26, except that, therein, the origin of the coordinate system lies at the midpoint of the generator of the cylindrical shell and the constants used are different.

Direct integration of Eq. (46), when the first of Eqs. (41) is used, yields the formula for u_s. Hence, u_s, the nondimensional axisymmetric static axial displacement, is

$$u_s(\xi, \tau) = \frac{\nu(l/a)}{\bar{K}} q_0(\tau) \left\{\xi + \frac{1}{2\mathcal{K}(\cos\mathcal{K} + \cosh\mathcal{K})} \times [\cos\mathcal{K}\xi \sinh\mathcal{K}(1 - \xi) + \sin\mathcal{K}(1 - \xi)\cosh\mathcal{K}\xi - \cos\mathcal{K}(1 - \xi)\sinh\mathcal{K}\xi - \sin\mathcal{K}\xi\cosh\mathcal{K}(1 - \xi)]\right\} \tag{47}$$

plus a constant.

Numerical Illustration

For a simple example, let one consider the case of a shell subjected on the outside to a semisinusoidal pulse of uniform pressure, the nondimensional quantity $q_0(\xi, \tau)$ of which, as defined in Eq. (2), may be represented below:

$$\left.\begin{array}{ll} q_0(\xi, \tau) = q_0(\tau) = A_0 \sin p\tau & 0 \le \tau \le (\pi/p) \\ q_0(\xi, \tau) = q_0(\tau) = 0 & (\pi/p) \le \tau \end{array}\right\} \tag{48}$$

where A_0 is a constant. It is shown diagrammatically in Fig. 2. It should be noted that, if the pressure is not uniform, q_m should replace q_0 and A_m should replace A_0 for $m \neq 0$.

Substitution of Eqs. (48) and (23a) in Eq. (18), in which q is replaced by q_0, yields the following:

$$P_i(\tau) = B_i \sin p\tau\{\mathfrak{U}(\tau) - \mathfrak{U}[\tau - (\pi/p)]\} \tag{49}$$

where

$$B_i = 2CA_0/Kk_i^2 n\pi \qquad (n = 1, 3, 5, \ldots) \tag{50}$$

and $\mathcal{U}(\tau)$ and $\mathcal{U}(\tau - \pi/p)$ are unit functions such that

$$\mathcal{U}(\tau) \triangleq \begin{cases} 0 & \text{when } \tau < 0 \\ 1 & \text{when } \tau \geq 0 \end{cases} \tag{51}$$

with the notation $\triangleq$ identifying "is defined to be." Equations (51) apply to $\mathcal{U}(\tau - \pi/p)$ as well if τ therein is only to be replaced by $\tau - \pi/p$.

Applying the Laplace transform to Eq. (21) after Eq. (49) is first introduced, one obtains

$$\mathcal{L}[\Phi_i(\tau)] = -\frac{B_i p}{m_i(s^2 + p^2)}(1 + e^{-\pi s/p})\left(1 - \frac{k_i^2}{s^2 + k_i^2}\right) \tag{52}$$

where s is the subsidiary parameter used for the transform. The result is

$$\Phi_i(\tau) = \frac{B_i p}{m_i(p^2 - k_i^2)}\left\{k_i \sin k_i\tau - p \sin p\tau + \left[k_i \sin k_i\left(\tau - \frac{\pi}{p}\right) - p \sin p\left(\tau - \frac{\pi}{p}\right)\right]\mathcal{U}\left(\tau - \frac{\pi}{p}\right)\right\} \tag{53}$$

or

$$\Phi_i(\tau) = \frac{B_i p}{m_i(p^2 - k_i^2)} \times (k_i \sin k_i\tau - p \sin p\tau) \qquad 0 \leq \tau \leq (\pi/p) \tag{54}$$

$$\Phi_i(\tau) = \frac{B_i p k_i}{m_i(p^2 - k_i^2)} \times [\sin k_i\tau + \sin k_i(\tau - \pi/p)] \qquad (\pi/p) \leq \tau \tag{55}$$

For static radial displacements, Eq. (46) may be used. To obtain the numerical results of the radial displacements of the exemplified case, a circular cylindrical shell with the following measurements is assumed: median radius a equal to 37.05 in., length l equal to 184.4 in., and thickness h equal to 0.4054 in. The properties of the material, which is taken to be stainless steel, may be represented by the constants $E = 29 \times 10^6$ psi, $\rho = 0.008609$ lb-sec^2/in.^{4}, and $\nu = \frac{1}{6}$. The choice of these numerical values is motivated by the desire to maintain the dimensionless ratios of Flügge's[7] example, which is described in metric units.

The maximum pressure of the imposed pulse is 1000 psi. Its circular frequency is assumed to be 1003 rad/sec, and consequently, the total duration of the semisinusoidal loading pulse is 3.133 msec. As a result, the nondimensional maximum pressure A_0 is equal to 1.884×10^5 and the nondimensional circular frequency p is equal to 3.14159.

The nondimensional natural frequencies for the axisymmetric case for $n = 1, 3, 5, \ldots, 19$ are listed in Table 2.

Figure 3 shows the response of the total radial displacements at the midpoint of the cylindrical shell where $\xi = 0.5$, as well as that of its component parts, namely, the static

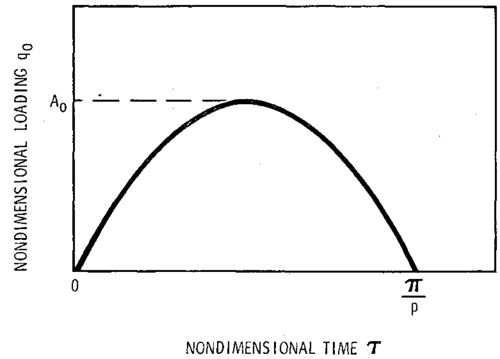

Fig. 2 Semisinusoidal pulse.

and the natural-mode displacements. The maximum radial displacement attained is $w = 56 \times 10^{-4}$ at $\tau = 0.8$ or $w^* = 0.21$ in. at $t = 2.5$ msec.

Stresses, Resultant Forces, and Moments

Since the nondimensional static as well as dynamic displacements in the radial, axial, and circumferential directions have all been derived, one is now in a position to formulate the stresses, resultant forces, and resultant moments as shown in Fig. 4, if Eqs. (7, 23, 26, and 27) are used, and Refs. 14 and 15 are consulted. The resulting formulas, which include both the axisymmetric and nonaxisymmetric portions, are listed below.

The resultant forces per unit length of the normal sections shown in Fig. 4b can be given as

$$N_x(\xi, \theta, \tau) = \frac{Eh}{1 - \nu^2}\sum_{m=0}^{\infty}\left\{\frac{u_s'(\xi, \tau)}{l/a} + \nu[m v_s(\xi, \tau) - w_s(\xi, \tau)] + 2C\sum_{n=1}^{\infty}\left[\sum_k \frac{\nu m^2(l/a)^2\beta_k - n^2\pi^2\alpha_k - (\nu/2)\gamma_k\eta_k}{\gamma_k\eta_k} \times \Phi_k(\tau)\right]\sin n\pi\xi\right\}\cos m\theta \tag{56a}$$

where the subscript k is introduced for clarity and stands for the two frequencies for each n when $m = 0$ and for the three frequencies for each n when $m \geq 1$:

$$N_\theta(\xi, \theta, \tau) = \frac{Eh}{1 - \nu^2}\sum_{m=0}^{\infty} \times \left\{\frac{\nu u_s'(\xi, \tau)}{l/a} + m v_s(\xi, \tau) - w_s(\xi, \tau) + 2C\sum_{n=1}^{\infty} \times \left[\sum_k \frac{m^2(l/a)^2\beta_k - \nu n^2\pi^2\alpha_k - (\frac{1}{2})\gamma_k\eta_k}{\gamma_k\eta_k}\Phi_k(\tau)\right]\sin n\pi\xi\right\}\cos m\theta \tag{56b}$$

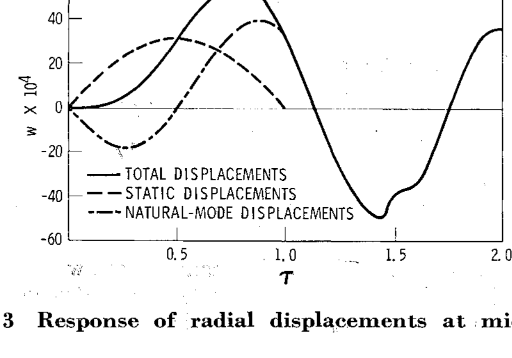

Fig. 3 Response of radial displacements at midspan.

Table 2 Nondimensional natural frequencies ($m = 0$)

n	k_i	
1	3.0703343	5.0207520
3	4.8814261	9.4745214
5	4.9017996	1.5732203 × 10
7	4.9127801	2.2007604 × 10
9	4.9308415	2.8286877 × 10
11	4.9636003	3.4567680 × 10
13	5.0187111	4.0849254 × 10
15	5.1044644	4.7131274 × 10
17	5.2295161	5.3413577 × 10
19	5.4023267	5.9696069 × 10

$$N_{x\theta}(\xi, \theta, \tau) = N_{\theta x}(\xi, \theta, \tau) = Eh \sum_{m=1}^{\infty} \times \left\{ \frac{1}{2(1+\nu)} \left[\frac{v_s'(\xi, \tau)}{l/a} - m u_s(\xi, \tau) \right] + m\left(\frac{l}{a}\right) \pi C \sum_{n=1}^{\infty} \times \left[\sum_k \frac{n(2n^2\pi^2 - \Omega k)}{\gamma_k \eta_k} \Phi_k(\tau) \right] \cos n\pi\xi \right\} \sin m\theta \quad (56c)$$

The moment resultants per unit length, indicated as right-hand vectors in Fig. 4c, are

$$M_x(\xi, \theta, \tau) = \frac{D/a}{(l/a)^2} \sum_{m=0}^{\infty} \left\{ \nu m^2 \left(\frac{l}{a}\right)^2 w_s(\xi, \tau) - w_s''(\xi, \tau) + C \sum_{n=1}^{\infty} \left(\left[n^2\pi^2 + \nu m^2 \left(\frac{l}{a}\right)^2 \right] \sum_k \Phi_k(\tau) \sin n\pi\xi \right) \right\} \cos m\theta \quad (57a)$$

$$M_\theta(\xi, \theta, \tau) = \frac{D/a}{(l/a)^2} \sum_{m=0}^{\infty} \left\{ m^2 \left(\frac{l}{a}\right)^2 w_s(\xi, \tau) - \nu w_s''(\xi, \tau) + C \sum_{n=1} \left(\left[m^2 \left(\frac{l}{a}\right)^2 + \nu n^2\pi^2 \right] \sum_k \Phi_k(\tau) \sin n\pi\xi \right) \right\} \cos m\theta \quad (57b)$$

$$M_{x\theta}(\xi, \theta, \tau) = -M_{\theta x}(\xi, \theta, \tau) = -(1-\nu)\left(\frac{D}{l}\right) \sum_{m=1}^{\infty} \left\{ m w_s'(\xi, \tau) + C\pi m \sum_{n=1}^{\infty} \left[(n \cos n\pi\xi) \sum_k \Phi_k(\tau) \right] \right\} \sin m\theta \quad (57c)$$

The shear forces per unit length, which accompany the moments indicated previously as well as those shown in Fig. 4b, can be represented as

$$Q_x(\xi, \theta, \tau) = \frac{Da}{l^3} \sum_{m=0}^{\infty} \left\{ m^2 \left(\frac{l}{a}\right)^2 w_s'(\xi, \tau) - w_s'''(\xi, \tau) + C\pi \sum_{n=1}^{\infty} \left[(n\Lambda \cos n\pi\xi) \sum_k \Phi_k(\tau) \right] \right\} \cos m\theta \quad (58a)$$

$$Q_\theta(\xi, \theta, \tau) = \frac{D}{l^2} \sum_{m=1}^{\infty} \left\{ m\, w_s''(\xi, \tau) - m^2 \left(\frac{l}{a}\right)^2 w_s(\xi, \tau) - C \sum_{n=1}^{\infty} \left[(\Lambda \sin n\pi\xi) \sum_k \Phi_k(\tau) \right] \right\} \sin m\theta \quad (58b)$$

As a consequence, the total effective shear resultants per unit length, as indicated in Fig. 4b, may be expressed as

$$Q_{x\cdot\mathrm{eff}}(\xi, \theta, \tau) = Q_x - \frac{\partial M_{x\theta}}{a\partial\theta} = \frac{Da}{l^3} \sum_{m=0}^{\infty} \left\{ (2-\nu) m^2 \left(\frac{l}{a}\right)^2 w_s'(\xi, \tau) - w_s'''(\xi, \tau) + C\pi \sum_{n=1}^{\infty} \left(n\left[\Lambda + (1+\nu) m^2 \left(\frac{l}{a}\right)^2 \right] \times \sum_k \Phi_k(\tau) \cos n\pi\xi \right) \right\} \cos m\theta \quad (59a)$$

$$Q_{x\cdot\mathrm{eff}}(\xi, \theta, \tau) = Q_\theta + \frac{\partial M_\theta}{a\partial\xi} = \frac{D}{l^2} \sum_{m=1}^{\infty} \left\{ (2-\nu) w_s''(\xi, \tau) - m^2 \left(\frac{l}{a}\right)^2 w_s(\xi, \tau) - C \sum_{n=1}^{\infty} \left([\Lambda + (1-\nu)n^2\pi^2] \sum_k \Phi_k(\tau) \sin n\pi\xi \right) \right\} \sin m\theta \quad (59b)$$

The arbitrary constant C will be canceled out when the generalized coordinate $\Phi_i(\tau)$ is evaluated and introduced after the pressure loading $q_m(\xi, \tau)$ is specified.

The normal stresses at extreme fibers of the cylindrical shell may be obtained by adding the membrane stresses and the bending stresses at that location, which are represented in the following:

$$\left.\begin{aligned} (\sigma_x)_{\text{extreme fiber}} &= (N_x/h) \pm (6/h^2) M_x \\ (\sigma_\theta)_{\text{extreme fiber}} &= (N_\theta/h) \pm (6/h^2) M_\theta \end{aligned}\right\} \quad (60)$$

Appendix

A. Rectangular Pulse Function

If the loading is of rectangular pulse form, Fig. 5, the following formulas may be useful:

$$\left.\begin{aligned} q_m &= A_m[\mathfrak{U}(\tau) - \mathfrak{U}(\tau - c)] \\ P_i(\tau) &= B_i[\mathfrak{U}(\tau) - \mathfrak{U}(\tau - c)] \\ \Phi_i(\tau) &= -(B_i/m_i)\{\cos k_i\tau - \mathfrak{U}(\tau - c)\cos[k_i(\tau - c)]\} \end{aligned}\right\} \quad (61)$$

B. Condition of Resonance

Whenever p, the nondimensional frequency of the imposed forcing function, happens to be very close to one of the natural frequencies of the cylindrical shell, i.e. $p \simeq k_i$, or letting $k_i = p + 2\Delta$, where Δ is a small quantity as compared with p, then

$$\Phi_i(\tau) \simeq -\frac{B_i}{2m_i(1 + \Delta/p)} \left[(1 - \Delta p\tau^2 - 2\Delta^2\tau^2) \sin p\tau + \left(1 - \frac{2}{3}\Delta^2\tau^2\right) p\tau \cos p\tau \right] \quad (62a)$$

when Δ's of third and higher orders are neglected. Conse-

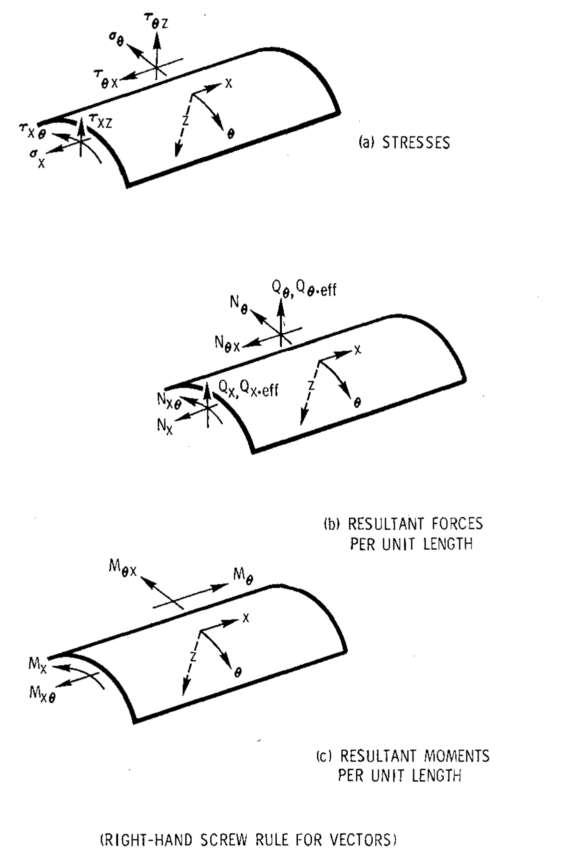

Fig. 4 Sign convention for stresses, forces, and moments.

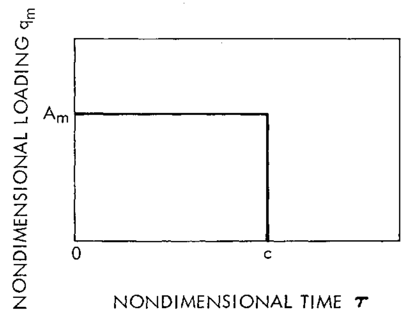

Fig. 5 Rectangular pulse.

quently, when the imposed frequency is identical with one of the natural frequencies, or $\Delta \equiv 0$, one gets

$$\Phi_i(\tau) = -\frac{B_i}{2m_i}(\sin p\tau + p\tau \cos p\tau) \qquad (62b)$$

which increases without bound as time goes on.

Concluding Remarks

This paper presents a method of analysis and the solution formulas in explicit form for the vibratory displacements and stresses of a thin circular cylindrical shell subjected to either steady or transient surface loading. It is emphasized that an increase of q raises the displacements and stresses proportionately, and that a change of $(E/\rho)^{1/2}$ extends or shortens the dimensional time and frequencies linearly. If the circular frequency of the imposed pulse is comparable to any of the low natural frequencies, Williams' method has a great advantage over the ordinary modal solutions, since the dynamic portions of the displacements during the loading pulse are expressed in a much faster convergent infinite series.

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